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EXACT CRITICAL VALUES OF THE KRUSKAL-WALLIS TEST

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ABSTRACT

Generating the entire permutation sample space has been identified as a major problem in constructing the exact test of significance of a rank statistic. Procedures for most of the existing methods for finding the exact distribution for the common test statistics that are in use today are usually based on asymptotics which only give approximate results. The definition of what constitute a large sample in order to apply the large sample approximation is quite vague. In this paper, a method for obtaining the exact distribution of the Kruskal-Wallis (K-W) test is presented. The proposed method is based on combinatorics in the representation of the probability generating function of the test statistic. Essentially, this paper produces the exact critical values of the K-W test and the minimum sample size required for the application of the large sample approximation of the K-W statistic.

Keywords: Exact test, rank test, permutation test, combinatorics, kruskal-wallis test.

INTRODUCTION

Exact Statistical method of data analysis is a valuable tool of applied statistics as it ensures that the probability of making a type I error is exactly α , thus controlling the risk in decision making. The idea of obtaining an exact test of significance through the permutation approach originated with Fisher (1935). Fisher compiled by hand 32,768 permutations of Charles Darwin's data on the height of cross-fertilized and self-fertilized zeamays plants. The enormity of this task possibly discouraged Fisher from probing further into exact permutation tests (Ludbrook and Dudley, 1998). In the quest for finding the exact distribution of several statistics and making correct inferences, a lot of problems abound. These problems are well highlighted in Fisher (1935), Agresti (1992), Mitic (1996), Baglivo *et al*. (1996), Bergmann *et al*. (2000), Good (2000), Odiase and Ogbonmwan (2005a,b) and Ogbonmwan *et al*. (2007).

Scheffe (1943) showed that the permutation approach is the only possible technique of constructing exact tests of significance for a general class of problems. Hoeffding (1952) remarked that this permutation test is asymptotically as powerful as the best parametric test. The unconditional permutation approach is a statistical procedure that ensures that the probability of a type I error is exactly α and ensures that the resulting distribution of the test statistic is exact, Agresti (1992), Good (2000), Pesarin (2001), Odiase and Ogbonmwan (2005, 2007). The unconditional exact permutation approach in which row and column totals are allowed to vary with each permutation is very much unlike the conditional exact permutation approach of fixing the row and column totals, Headrick (2003), Bagui and Bagui (2004) and Odiase and Ogbonmwan (2005b). Exact tests constructed by restricting attention to a conditional reference set of contingency tables with margins fixed at the values actually observed is not always true in nature.

There are several Monte Carlo methods that can be used in generating exact p-values. The most widely used is the bootstrap re-sampling technique developed by Efron (1979). The Bayesian and the Likelihood approaches can be found in Bayarri and Berger (2004) and Spiegelhalter (2004). All these alternative approaches to the unconditional permutation approach only give approximate results. Exact procedures are the best and should always be applied whenever practically possible, Lehmann (1986) and Good (2000). Permutation tests provide exact results especially when complete enumeration is possible (Pesarin, 2001).

A complete enumeration of the permutation sample space for the purpose of constructing an exact test of significance is only possible when sample sizes are small (Odiase and Ogbonmwan, 2005a,b). A big challenge in using nonparametric test is the availability of computational formulas and tables of exact critical values. This continues to be a problem as evidenced by a survey of 20 in-print general college statistics texts, see Fahoome (2002).

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Many nonparametric tests have large sample approximations that can be used as an alternative to tabulated exact critical values. These approximations are useful substitutes if the sample size is sufficiently large and hence obviate the need for locating tables of exact critical values. But, the definition of what constitutes a large sample size for most statistics is quite vague in the literature (Bergmann *et al*., 2000; Fahoome, 2002). This paper produces the exact critical values of the K-W statistic. The consideration given here overcomes the major problem of carrying out a complete enumeration in order to construct an exact test of significance. The exact distribution of the K-W statistic is obtained in a combinatorial sense via its probability generating function using the computer algebra Mathematica 6.0. When sample sizes are large, the exact distribution of the K-W statistic can be approximated by the chi square distribution. We study the convergence of the chi square approximation to the exact distribution of the K-W statistic and provide the minimum sample size required for the application of this asymptotic distribution.

Exact distribution of the Kruskal-Wallis test

Kruskal and Wallis (1952) rank based test of location equality for $k \geq 3$ may be among the most useful of available hypothesis testing procedures. The K-W statistic can be seen as a generalization of the Wilcoxon rank sum test. Consider k independent samples $X_{11},\ldots,X_{1n_1}$, $X_{21},\ldots,X_{2n_2}$, $\ldots,X_{k1},\ldots,X_{kn_k}$ of sizes $n_1,\ldots,n_k$ drawn from k continuous (not necessarily normal) populations. We wish to test the null hypothesis H_o that these populations are identically distributed against the alternative H_1 that these populations are not identically distributed.

To conduct the test, we calculate the rank sum $R_1,\ldots,R_k$ of the $X_{1.}'s,\ldots,X_{k.}'s$ in the combined ordered arrangement of these k samples and the K-W test statistic is

$$H = \textit{K-W[nlist]} = \frac{12}{n(n+1)}\sum_{i=1}^{k}\frac{R_i^2}{n_i} - 3(n+1) \qquad (1)$$

with $n = \sum_{i=1}^{k} n_i$ and *[nlist]* is a list of the sample sizes.

The null hypothesis H_o is rejected if $H \geq H_{1-\alpha}$ where the notation $H_{1-\alpha}$ in Tables 1 through 3 means $(1-\alpha)100\%$ percentile of the *H* statistic which is equivalent to α level exact critical value of the *H* statistic. A direct enumeration of all the permutation of ranks in a k sample experiment is required to compute the exact distribution of the K-W statistic. However, this is only feasible for very small sample sizes. Problem arises as the sample size increases. For instance, to compute the exact distribution of K-W statistic for the sample sizes $n_1 = n_2 = n_3 = 10$, we require 5,550,996,791,340 distinct arrangements of the ranks. To overcome this problem of complete enumeration, we state a combinatorial problem following the idea of Baglivo *et al*. (1996). We develop generating functions to solve the problem formulated. This provides insight into obtaining the exact distribution of the K-W statistic.

Combinatorial Problem

We pose the following combinatorial problem: Suppose we have n observations which are ranked 1, 2, 3…, n. In how many different ways is it possible to divide these n observations among k samples such that the i^{th} sample T_i contains n_i observations and the sum of the ranks of these n_i observations in sample T_i is r_i with

$$n = \sum_{i=1}^{k} n_i \text{ and } r = \sum_{i=1}^{k} r_i = \tfrac{1}{2}n(n+1)?$$

Let the number be:

$$P[nlist, rlist] := P[\{n_1, n_2, \ldots, n_k\}, \{r_1, r_2 \ldots, r_k\}] \qquad (2)$$

We can calculate this number $P[nlist, rlist]$ by counting the relevant partitions.

There are $\dfrac{(n_1 + n_2 + \ldots + n_k)!}{n_1! n_2! \ldots, n_k!}$

possible permutations of the n variates of the k samples of sizes $n_i, i = 1,2,\ldots,k$ which are equally likely with probability $\left(\dfrac{n!}{n_1! n_2! \ldots, n_k!}\right)^{-1}$.

The number $P[nlist, rlist]$ can easily be obtained for small n and k by counting the relevant partitions.

However, when n and k are small, this method of obtaining $P[nlist, rlist]$ fails because of the large associated permutation sample spaces. For instance, when $n_1 = 6, n_2 = 7, n_3 = 4, n_4 = 3$, there are 4,655,851,200 distinct arrangements of the ranks. Admittedly, it is very difficult to carry out this enumeration manually in order to compute $P[nlist, rlist]$.

To overcome this major problem of enumeration, we find the generating function for the number $P[nlist, rlist]$. To achieve this, let $x[i]$ be a variable governing the number of observations in the i^{th} sample and $y[i]$ be a variable governing the sum of the ranks of the observations in the i^{th} sample. Then, the generating function for the number $P[nlist, rlist]$ is given as

$$p[n,k] = \prod_{j=1}^{n}\left(\sum_{i=1}^{k} x[i]y[i]^{j}\right) \tag{3}$$

See Ewere and Ogbonmwan (2010a)

Obviously, the numbers $P[nlist, rlist]$ are the coefficients of $\prod_{i=1}^{k} x[i]^{n_i} y[i]^{r_i}$ of the polynomial $p[n,k]$. Hence, $P[nlist, rlist]$ is obtained by selecting the coefficients of $\prod_{i=1}^{k} x[i]^{n_i} y[i]^{r_i}$. However, this method of enumeration is not computationally efficient as one would expect due to the fact that the number of terms of the generating function in (3) are of order k^n which is not too small even if n and k are not very large.

To improve on the computational efficiency of (3), we let $nlist = \{n_1, n_2, \ldots, n_k\}$. This gives rise to a new generating function $p[nlist]$ for the number $P[nlist, rlist]$ which have number of terms whose order is only $Multinomial[n_1, n_2, \ldots, n_k]$ which is smaller than k^n. Clearly, this generating function $p[nlist]$ gives the coefficients of $\prod_{i=1}^{k} x[i]^{n_i}$ of the generating function $p[n,k]$. To speed up computations, the generating function $p[nlist]$ is defined recursively as:

$$p[nlist] = p[\{n_1, n_2, \ldots n_k\}] = \sum_{i=1}^{k} y[i]^{n} p[\{n_1, n_2, \ldots n_i - 1, \ldots n_k\}] \tag{4}$$

See Ewere and Ogbonmwan (2010b).

To compute the exact null distribution of the K-W statistic, It is instructive to state that:'

$$\Pr ob(\{R_1 = r_1\} \cap \{R_2 = r_2\} \cap \ldots \cap \{R_k = r_k\}) = \frac{P[nlist, rlist]}{Multinomial[n_1, \ldots, n_k]} \tag{5}$$

where

$$Multinomial[n_1, \ldots, n_k] = \frac{\left(\sum_{i=1}^{k} n_i\right)!}{\prod_{i=1}^{k} n_i!} = \mathrm{M} \tag{6}$$

The generating function of the K-W statistic is obtained by substituting H in (1) to

$$\frac{p[nlist]}{Multinomial[n_1, \ldots, n_k]} \tag{7}$$

From this generating function, all the distributional characteristics of the K-W statistic can be obtained. Under the null hypothesis, all possible assignments of the ranks 1,2,…, n to the k samples are equally likely with probability M^{-1}.

The large sample approximation

Good (2000) noted that when sample sizes are large, the time required to compute a permutation distribution can be prohibitive even if we are taking advantage of one of the optimal computing algorithm. However, when sample sizes are large, asymptotic approximations are often used in place of the exact permutation distribution. Nevertheless, these asymptotic approximations are to be avoided except with very large samples as they can be grossly in error, see Micceri (1989), Mudholkar and Hutson (1997) and Good (2000). There is no general agreement in the literature as to what constitute a large sample for several statistics, see Fahoome (2002).

Ascertaining the smallest sample size that can be used with a large sample approximation for various statistics would enable researchers who do not have access to the necessary tables of critical values to employ these tests. In order to determine the minimum sample size required for the application of asymptotic result of the K-W statistic, we employed the Bradley's (1978) conservative estimates of 0.045 < Type I error rate < 0.055 and 0.009 < Type I error rate < 0.011 as measures of robustness when nominal α was set at 0.05 and 0.01 respectively. The sample sizes were increased until the Type I error rates converged within these acceptable regions. Generally, the stringent criterion $0.9\alpha \le \alpha_0 \le 1.1\alpha$ where α_0 is the true probability of a type I error when one or more of a test's assumptions are violated and the null hypothesis is true seems more appropriate to illustrations of 'convergence' than the liberal criterion given by Cochran (1952), who considered actual significance levels less than 20% above the nominal level to be acceptable, see Sullivan and D'Agostino (1992).

RESULTS AND DISCUSSION

Tables 1, 2 and 3 provide exact critical values for the K-W statistic. These exact critical values ensures that the probability of a type I error in decision making arising from the use of the K-W test is exactly α . These critical values have been presented for some combination of sample sizes as it is impractical to present the entire distribution here. But, this can be easily obtained from the Mathematica program. For large sample sizes, the distribution of the K-W test statistic can be approximated by a chi-square distribution. We study the quality and usefulness of this approximation both numerically and graphically. We reported the exact and asymptotic type I error rates for nominal level of significance $\alpha = 0.10$ and 0.05 when k = 3 and $\alpha = 0.10$ for a 4 sample situation, that is k = 4, see Tables 4, 5 and 6 respectively.

We were restricted to these significance levels due to limitation in computer memory. Using the Bradley's (1978) measure of robustness, we found that a minimum of 3 and 9 per sample for k = 3 is adequate to apply the chi- Square distribution for $\alpha = 0.10$ and $\alpha = 0.05$ respectively. For a 4 sample condition, the minimum size per sample is 4 for the application of the Chi-Square distribution when $\alpha = 0.10$. These results have been asterisked in Tables 4 through 6. Tables 4 through 6 reveals the convergence of the asymptotic distribution to the exact distribution of the K-W statistic as the sample size increases. This is evident in the decrease in the absolute difference between the exact and asymptotic Type I error rates as the sample size increases. Figures 1 through 4 also shows this convergence situation.

The definition of a large sample in order to apply the asymptotic result of the K-W test is quite vague in the literature. Kruskal and Wallis (1952) found that for small α (less than about 0.10) and k = 3, the Chi-square approximation furnishes a conservative test in many if not most situations. Gabriel and Lachenbruch (1969) showed that the Chi-square approximation is good even though the sample sizes may be small. Conover (1971) concluded that the Chi-square approximation should be used when the sample sizes exceed 5. He further stated that though the Chi-square approximation is justified only for reasonably large sample sizes n_i, in practice the approximation is used in all situations not covered by the table provided by Kruskal and Wallis (1952). Devore (1982) and Rohatgi (1984) stated that the large sample approximation is applied if k = 3, $n_i \ge 6, i = 1(1)3$ or k > 3, $n_i \ge 5, i = 1(1)k$. Fahoome (2002) recommended a minimum of 11 and 22 per sample for $\alpha = 0.05$ and $\alpha = 0.01$ for the application of the Chi-square approximation. Bagui and Bagui (2004) noted that for small samples n_i, $i = 1(1)k$ in a k–sample experiment, the null distribution of K-W statistic is not known and a Chi-square approximation will not be a good one. These recommendations in the literature for the minimum sample sizes for the application of the asymptotic result of the K-W statistic were based on approximate methods and thus not reliable.

Table 1. Exact Critical Values of the Kruskal-Wallis test for k = 3

Sample sizes	$H_{0.9000}$	$H_{0.9500}$	$H_{0.9750}$	$H_{0.9900}$	$H_{0.9950}$	$H_{0.9975}$	$H_{0.9990}$
2, 2, 2	3.71429	4.57143	-	-	-	-	-
3, 2, 2	4.46429	4.5000	5.35714	-	-	-	-
3, 3, 2	4.55556	5.13889	5.55556	6.2500	-	-	-
3, 3, 3	4.62222	5.6000	5.95556	6.48889	6.48889	7.2000	-
4, 2, 2	4.45833	5.1250	5.33333	6.0000	-	-	-

4, 3, 2	4.4444	5.4000	5.8000	6.3000	6.4000	7.0000	-
4, 3, 3	4.70909	5.79091	6.01818	6.74545	7.0000	7.31818	8.01818
4, 4, 2	4.44545	5.23636	6.08182	6.87273	7.03636	7.85455	-
4, 4, 3	4.47727	5.57576	6.38636	7.13636	7.47727	7.84848	8.32576
4, 4, 4	4.5000	5.65385	6.57692	7.53846	7.73077	8.11538	8.76923
5, 2, 2	4.29333	5.04000	5.69333	6.13333	6.53333	-	-
5, 3, 2	4.49455	5.10545	5.94909	6.82182	6.94909	7.18182	7.63636
5, 3, 3	4.41212	5.51515	6.30303	6.98182	7.51515	7.87879	8.24242
5, 4, 2	4.51818	5.26818	6.04091	7.11818	7.56818	7.81364	8.11364
5, 4, 3	4.52308	5.63077	6.39487	7.39487	7.90641	8.25641	8.62564
5, 4, 4	4.61868	5.61758	6.5967	7.74396	8.15604	8.7033	9.12857
5, 5, 2	4.50769	5.24615	6.23077	7.26923	8.07692	8.29231	8.68462
5, 5, 3	4.53626	5.62637	6.48791	7.54286	8.26374	8.65934	9.05495
5, 5, 4	4.5200	5.64286	6.67143	7.79143	8.46286	9.02571	9.50671
5, 5, 5	4.5000	5.6600	6.7200	7.9800	8.7200	9.3800	9.7800
6, 2, 2	4.43636	5.01818	5.52727	6.54545	6.65455	6.98182	-
6, 3, 2	4.54545	5.22727	6.06061	6.72727	7.5000	7.57576	8.18182
6, 3, 3	4.53846	5.55128	6.38462	7.19231	7.61538	8.32051	8.62821
6, 4, 2	4.4359	5.26282	6.10897	7.21154	7.82051	8.30769	8.66667
6, 4, 3	4.5989	5.6044	6.5000	7.46703	8.02747	8.61538	9.15385
6, 4, 4	4.52381	5.66667	6.59524	7.72381	8.32381	8.88095	9.62857
6, 5, 2	4.47473	5.31868	6.18901	7.2989	8.18681	8.74725	9.18462
6, 5, 3	4.49714	5.6000	6.62095	7.5600	8.29714	9.02857	9.61714
6, 5, 4	4.5000	5.65583	6.73583	7.89583	8.6400	9.29333	9.9600
6, 5, 5	4.52941	5.69853	6.78088	8.01176	8.83529	9.58088	10.2706
6, 6, 2	4.41905	5.35238	6.17143	7.40952	8.15238	8.93333	9.67619
6, 6, 3	4.5250	5.6000	6.68333	7.68333	8.41667	9.2250	10.1250
6, 6, 4	4.51838	5.72059	6.78309	7.98897	8.72059	9.41176	10.2831
6, 6, 5	4.54118	5.75163	6.83791	8.11895	8.9817	9.72418	10.5150
6, 6, 6	4.53801	5.7193	6.87719	8.18713	9.08772	9.87135	10.8421
7, 7, 7	4.54917	5.76623	6.90909	8.33395	9.35807	10.2486	11.3098
8, 8, 8	4.5800	5.7950	6.9800	8.4350	9.4850	10.5000	11.6550
9, 9, 9	4.57496	5.82363	7.02998	8.55379	9.59788	10.6667	11.9189
10,10,10	4.5600	5.8529	7.07871	8.6271	9.75742	10.7845	12.1110
11,11,11	4.58143	5.84735	7.11911	8.67088	9.81429	10.9110	12.3034
12,12,12	4.58258	5.86937	7.11261	8.72222	9.87538	11.0045	12.4339
13,13,13	4.58107	5.87337	7.13964	8.75621	9.94201	11.0923	12.5550
14,14,14	4.57997	5.89559	7.15425	8.7907	9.98481	11.1467	12.6246
15,15,15	4.58512	5.90609	7.17527	8.81391	10.0321	11.2139	12.7173

Table 2. Exact Critical Values of the Kruskal-Wallis test for k = 4

Sample sizes	$H_{0.9000}$	$H_{0.9500}$	$H_{0.9750}$	$H_{0.9900}$	$H_{0.9950}$	$H_{0.9975}$	$H_{0.9990}$
2,2,2,2	5.5000	6.0000	6.16667	6.16667	-	-	-
3,2,2,2	5.64444	6.24444	6.64444	7.0000	7.13333	-	-
3,3,2,2	5.72727	6.47273	7.0000	7.63636	7.72727	8.0000	8.12727
3,3,3,2	5.81818	6.68182	7.4697	7.95455	8.31818	8.56061	8.92424
3,3,3,3	5.97436	6.89744	7.61538	8.4359	8.74359	9.15385	9.46154
4,2,2,2	5.67273	6.43636	6.98182	7.30909	7.85455	7.96364	-
4,3,2,2	5.71212	6.61364	7.31818	7.84848	8.2500	8.59091	8.89394
4,3,3,2	5.85897	6.78205	7.55769	8.32051	8.69872	9.05769	9.40385
4,3,3,3	6.0000	6.96703	7.75824	8.65385	9.23077	9.57692	10.000
4,4,3,3	6.00476	7.03333	7.92381	8.86667	9.49048	9.96667	10.4619
4,4,4,3	6.02917	7.12917	8.05417	9.06667	9.71667	10.3417	10.900
4,4,4,4	6.06618	7.21324	8.20588	9.26471	9.94853	10.5662	11.3382
5,2,2,2	5.61818	6.52727	7.15455	7.66364	8.01818	8.38182	8.68182
5,3,2,2	5.75385	6.65641	7.4641	8.19487	8.62564	8.93333	9.42308
5,3,3,2	5.85714	6.82198	7.65055	8.59121	9.05714	9.41758	9.8549
5,3,3,3	5.9981	7.01143	7.82667	8.8400	9.45714	9.90667	10.4095
5,4,3,3	6.02917	7.08917	7.98917	9.02917	9.69583	10.2892	10.8558
5,4,4,3	6.03456	7.16691	8.13456	9.21176	9.93971	10.5574	11.2963
5,4,4,4	6.06078	7.25686	8.27255	9.3902	10.1373	10.8020	11.5882
5,5,4,4	6.06842	7.28947	8.34211	9.53509	10.3281	11.0228	11.8439
5,5,5,4	6.07737	7.32632	8.40632	9.66474	10.4858	11.2232	12.0947
5,5,5,5	6.09714	7.36571	8.47429	9.78857	10.6571	11.4114	12.3143
6,2,2,2	5.66667	6.51282	7.28205	7.89744	8.38462	8.66667	9.23077
6,3,2,2	5.76923	6.69231	7.48352	8.35165	8.84615	9.24176	9.72527
6,3,3,2	5.87619	6.85714	7.69524	8.68571	9.30476	9.7619	10.2190
6,3,3,3	6.0250	7.03333	7.89167	8.9000	9.6000	10.2250	10.7583
6,4,3,3	6.02206	7.1250	8.04779	9.1250	9.84926	10.5147	11.2059
6,4,4,3	6.04902	7.18954	8.18301	9.31373	10.0686	10.7582	11.5523
6,4,4,4	6.06725	7.26901	8.32164	9.4883	10.2632	10.9883	11.8392
6,5,4,4	6.07737	7.31421	8.39368	9.61947	10.4437	11.1937	12.0826
6,5,5,4	6.09048	7.3400	8.4619	9.7519	10.6157	11.3857	12.3048
6,5,5,5	6.10823	7.37316	8.51861	9.86926	10.7602	11.5636	12.5186
6,6,5,5	6.0996	7.4000	8.56522	9.94545	10.8743	11.7091	12.7075
6,6,6,5	6.10942	7.14377	8.60217	10.0152	10.9790	11.8507	12.8942

Table 3. Exact Critical Values of the Kruskal-Wallis test for k = 5

Sample sizes	$H_{0.9000}$	$H_{0.9500}$	$H_{0.9750}$	$H_{0.9900}$	$H_{0.9950}$	$H_{0.9975}$	$H_{0.9990}$
2,2,2,2,2	6.87273	7.30909	7.85455	8.07273	8.4000	8.40000	-
2,3,2,2,2	6.93939	7.66667	8.16667	8.66667	8.95455	9.0303	9.27273
2,3,3,2,2	7.01282	7.89744	8.52564	9.10256	9.44872	9.69231	10.000
2,3,3,3,2	7.0989	8.02198	8.8022	9.49451	9.86813	10.2198	10.6044
2,3,3,3,3	7.18095	8.17143	9.00952	9.84762	10.3048	10.6857	11.1429
3,3,3,3,3	7.3000	8.3000	9.2000	10.1667	10.7000	11.1333	11.6330
4,2,2,2,2	6.98077	7.82692	8.38462	9.05769	9.36538	9.59615	9.92308
4,3,2,2,2	7.04396	7.97802	8.7033	9.41758	9.85714	10.2033	10.5385
4,3,3,2,2	7.1381	8.12381	8.9381	9.78095	10.2619	10.6762	11.1000
4,3,3,3,2	7.2375	8.2625	9.1375	10.0917	10.6500	11.1042	11.5958

Table 4. Exact and Asymptotic Type I error rates for K-W $\alpha = 0.10$ and k = 3

Sample sizes	**Exact**	**Asymptotic**	**|Exact – Asymptotic|**
2, 2, 2	0.066667	0.157237	0.09057
* 3, 3, 3*	0.085714	0.0991511	0.0134371
4, 4, 4	0.096623	0.105399	0.008776
5, 5, 5	0.099520	0.105399	0.005879
6, 6, 6	0.098737	0.103415	0.004678
7, 7, 7	0.099327	0.102840	0.003513
8, 8, 8	0.099331	0.101266	0.001935
9, 9, 9	0.099584	0.101522	0.001938
10, 10, 10	0.099717	0.102284	0.002567
11, 11, 11	0.099340	0.101194	0.001854
12, 12, 12	0.099583	0.101136	0.001553
13, 13, 13	0.099826	0.101212	0.001386
14, 14, 14	0.099902	0.101268	0.001366
15, 15, 15	0.099967	0.101008	0.001041

* Minimum sample size for the application of the asymptotic result of the K-W

Table 5. Exact and Asymptotic Type I error rates for K-W $\alpha = 0.05$ and k = 3

Sample sizes	**Exact**	**Asymptotic**	**|Exact– Asymptotic|**
3, 3, 3	0.028571	0.0608101	0.0322391
4, 4, 4	0.048658	0.0591946	0.0105366
5, 5, 5	0.048777	0.0590129	0.0102359
6, 6, 6	0.049054	0.0572888	0.0082348
7, 7, 7	0.049108	0.0559602	0.0068522
8, 8, 8	0.049733	0.0551610	0.005428
* 9, 9, 9*	0.049946	0.0543769	0.0043309
10, 10, 10	0.049897	0.0535869	0.0036899
11, 11, 11	0.049852	0.0537358	0.0038838
12, 12, 12	0.049969	0.0531475	0.0031785
13, 13, 13	0.049987	0.0530413	0.0030543
14, 14, 14	0.049968	0.0524552	0.0024872
15, 15, 15	0.049934	0.0521806	0.0022466

* Minimum sample size for the application of the asymptotic result of the K-W.

Table 6. Exact and Asymptotic Type I error rates for K-W $\alpha = 0.10$ and k = 4

Sample sizes	**Exact**	**Asymptotic**	**|Exact– Asymptotic|**
2, 2, 2, 2	0.076190	0.138639	0.062449
3, 3, 3, 3	0.097792	0.112864	0.015072
* 4, 4, 4, 4*	0.099001	0.108434	0.009433
5, 5, 5, 5	0.099340	0.106406	0.007066

* Minimum sample size for the application of the asymptotic result of the K-W

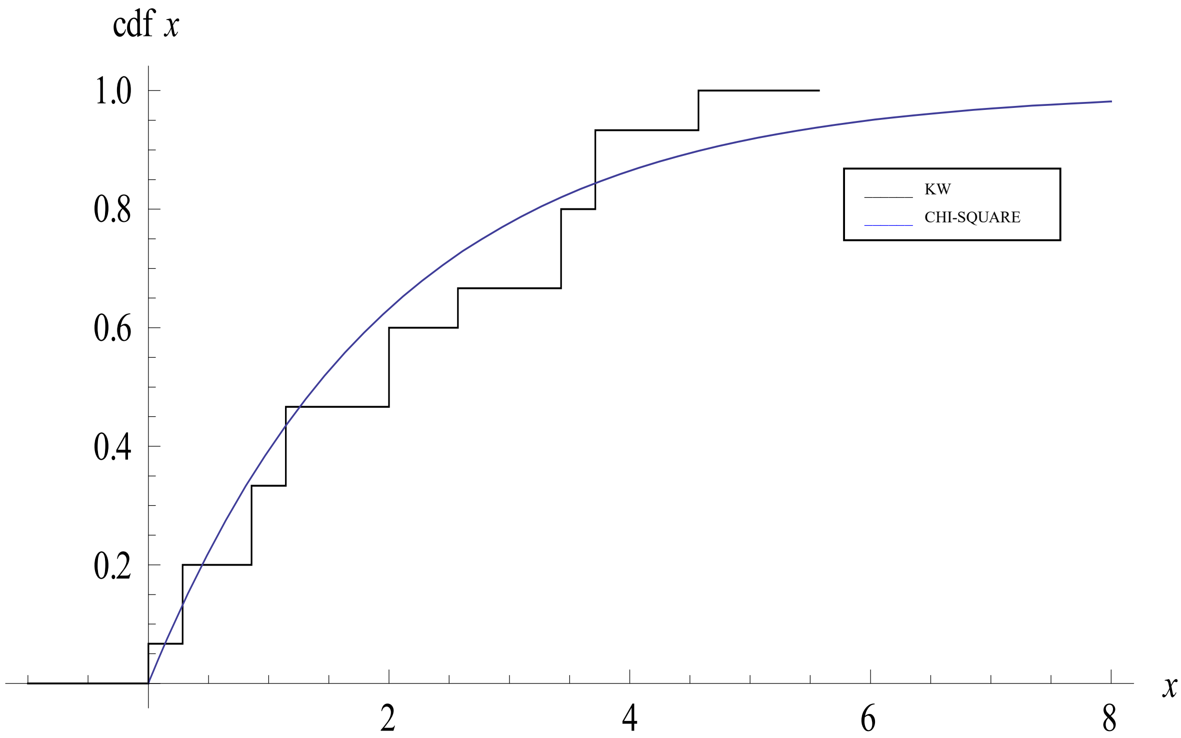

Fig. 1. Exact and asymptotic cumulative distribution functions of the K-W when $n_1 = n_2 = n_3 = 2$

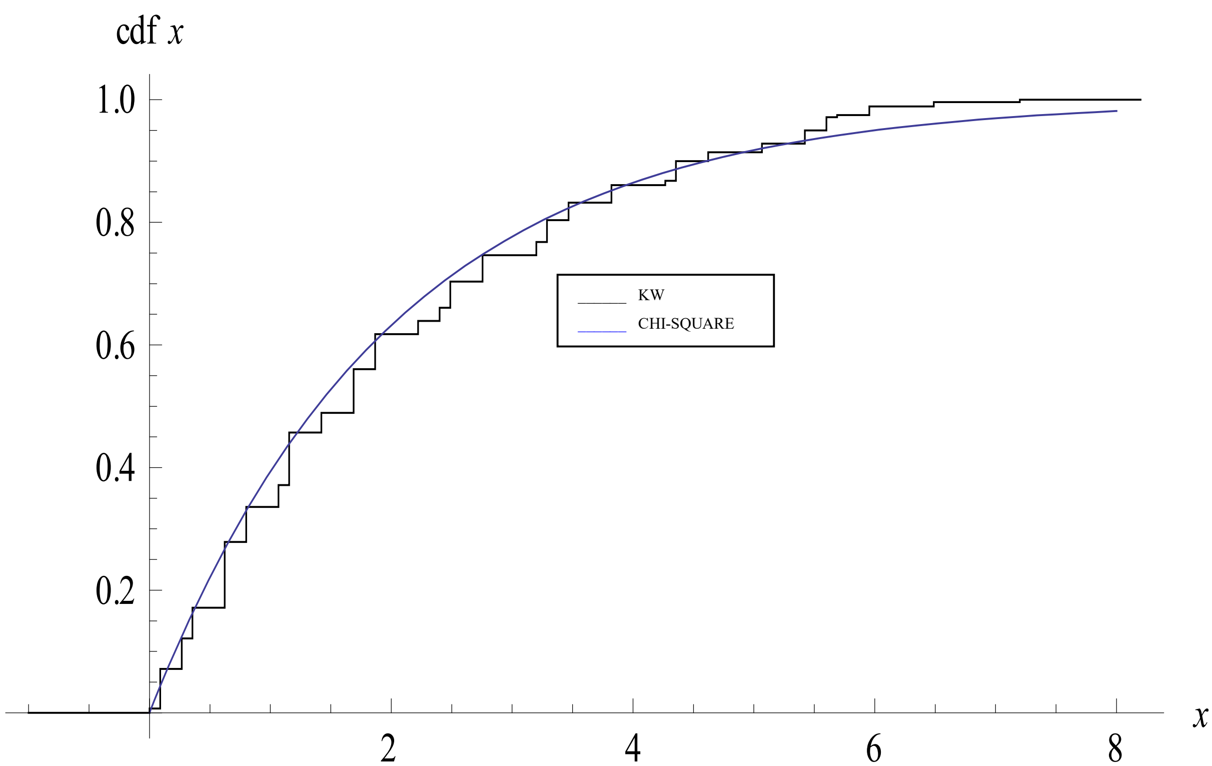

Fig. 2. Exact and asymptotic cumulative distribution functions of the K-W when $n_1 = n_2 = n_3 = 3$

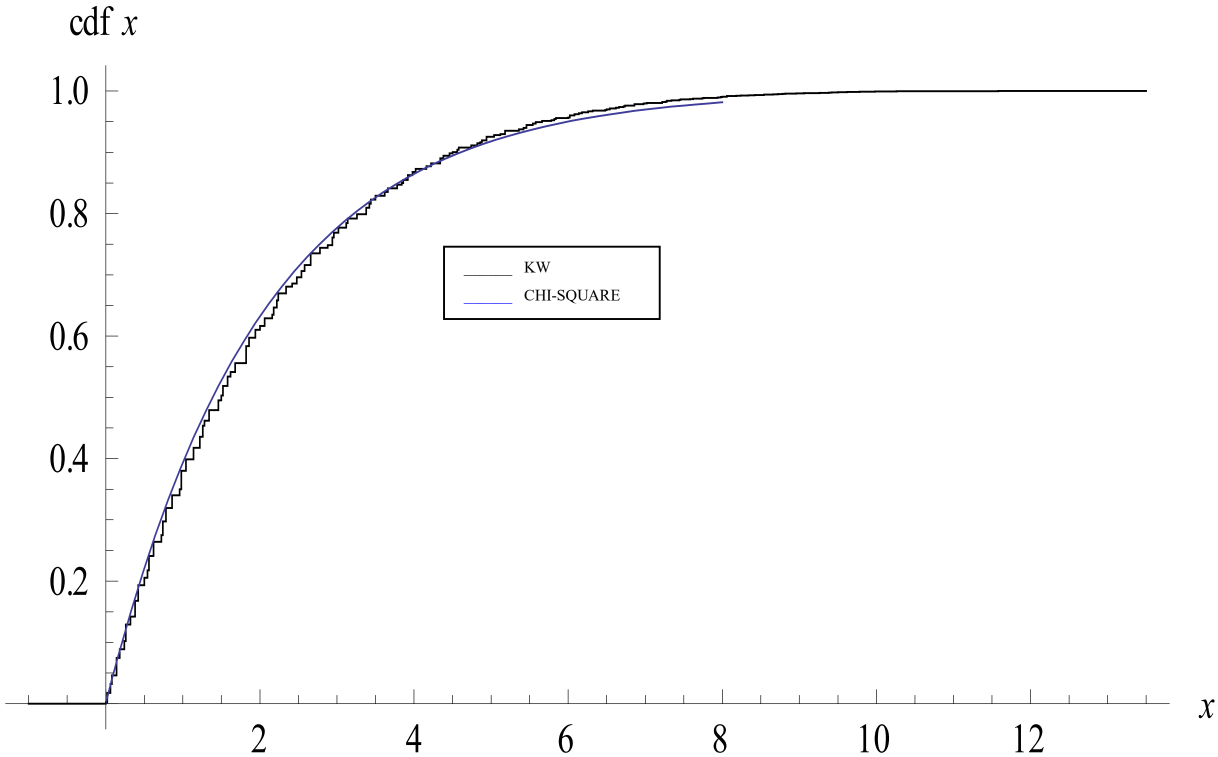

Fig. 3. Exact and asymptotic cumulative distribution functions of the K-W when $n_1 = n_2 = n_3 = 5$

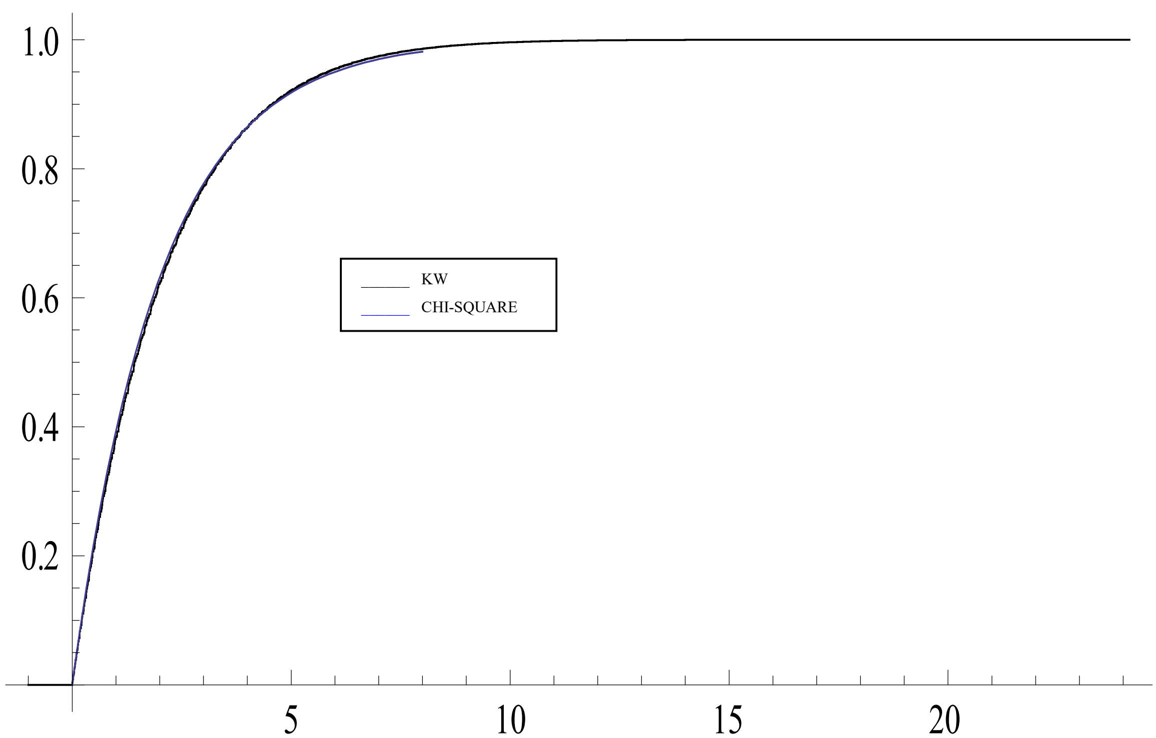

Fig. 4. Exact and asymptotic cumulative distribution functions of the K-W when $n_1 = n_2 = n_3 = 9$

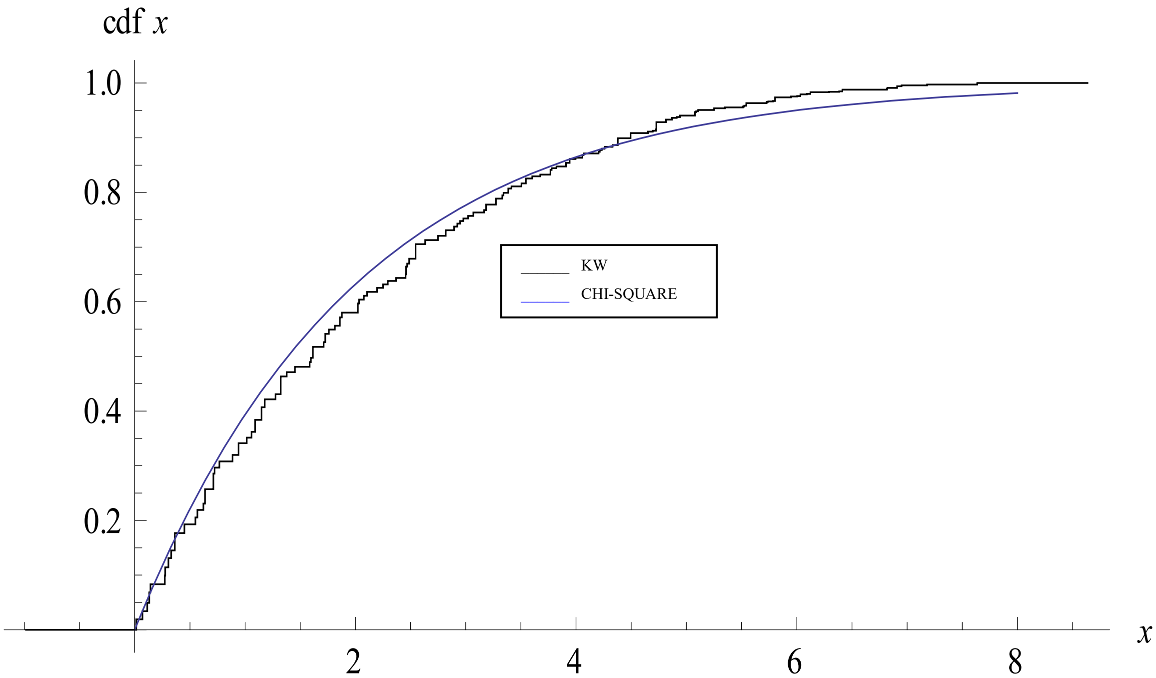

Fig. 5. Exact and asymptotic cumulative distribution functions of the K-W when $n_1 = 2, n_2 = 3, n_3 = 5$

CONCLUSION

This paper provides a sure and efficient method for obtaining the exact distribution of the K-W statistic and the proposed method overcomes the major problem of carrying out a complete enumeration in order to construct the exact test of significance. The exact critical values for the K-W statistic have been produced and the minimum sample size required for the application of the asymptotic distribution has been presented.

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